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The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation

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Abstract

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We construct a combinatorial CW-complex KP_n whose vertices correspond to all possible bracketings of all possible permutations of n letters $A_1, \ldots, A_n$. This structure is implicitly present in Mac Lane's coherence theorem for symmetric and braided monoidal categories. It also fits very naturally in the framework of the study of Knizhnik–Zamolodchikov (KZ) equations initiated by V. Drinfel'd. We show that KP_n is a combinatorial $(n-1)$-ball and establish its connection with the Grothendieck–Knudsen moduli space of stable n-pointed curves of genus 0.

Introduction

The purpose of this paper is to construct and study a certain 'hybrid' of two polytopes important in the theory of monoidal categories and H-spaces: the associahedron (or Stasheff polytope) K_n (see [30]) and the permutohedron (or the general hypersimplex) P_n (see [1, 11, 23]). Probably the simplest motivation for the study of K_n and P_n is provided by Mac Lane's coherence theorem for associativities and commutativities in monoidal categories [21].

Namely, vertices of the polytope K_n correspond to all bracketings of n letters. This polytope was constructed first by Stasheff [30] as a CW-ball (and even as a convex body in $\mathbb{R}^{n-2}$) and was given a realization as a convex polytope (i.e. a

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convex hull of a finite set of points in $\mathbb{R}^{n-2}$) in the papers [10, 20] and also earlier by Milnor (unpublished). Given any n objects $A_1, \ldots, A_n$ of a monoidal category, the associativity isomorphisms give a diagram whose shape is the 1-skeleton of K_n (so its vertices are all the bracketed products of A_i in the given order). The coherence conditions for associativity given by Mac Lane amount to the commutativity of each 2-face of K_n, thereby giving the commutativity of the whole diagram (since K_n is a ball).

The permutohedron P_n is, by definition, the convex hull of a generic orbit of the symmetric group S_n in $\mathbb{R}^n$, see [1, 11, 23] (the name is borrowed from [18]). Its vertices correspond to all permutations of n letters. Therefore, if we have a monoidal category with *strict* associativity and, in addition, commutativity data $R_{A,B}: A \otimes B \rightarrow B \otimes A$ (satisfying natural axioms) then for any n objects $A_1, \ldots, A_n$ these data furnish a diagram of the shape P_n (whose vertices are all the permuted products of A_i). Again, the coherence conditions imply the commutativity of each 2-face of this diagram thereby proving the commutativity of the whole P_n (i.e., the coherence theorem).

It is natural, therefore, to look for a 'polytope' KP_n whose vertices would correspond to all bracketed and permuted products of n letters so that any n objects in any symmetric (or braided) monoidal category give rise to a diagram of the shape KP_n. We shall construct KP_n, as a CW-complex, in Section 2 and show that it is an $(n-1)$-ball. This gives an instant one-step proof of Mac Lane's theorem in full generality. However, it remains unclear whether KP_n can be realized as a convex polytope, like K_n and P_n.

Our work on this problem was motivated by, besides its beauty and naturality, potential applications in the theory of Yang–Baxter equations and their higher-dimensional generalizations. This theory uses two essentially equivalent languages: Hopf algebras and monoidal categories. The study of various 'weak' conditions, traditional for categories, was undertaken in the context of Hopf algebras by Drinfel'd [7, 8], who considered so-called quasi-Hopf algebras (where the coassociativity axiom is replaced by a certain weak version). More recently, Stasheff and Takhtajan [31], in order to study cocommutative quasi-Hopf algebras, constructed a 3-dimensional polytope whose vertices correspond to permuted bracketed products of four *commuting* letters. Their polytope can be seen as a coarsening of our KP_4. To give a general definition of a quasi-triangular quasi-Hopf algebra one needs the full KP_4. On the categorical side, in [15] a categorical interpretation of Zamolodchikov's tetrahedra equations [32] (a 2 + 1-dimensional generalization of the Yang–Baxter equations) is given. It involves the commutativity of a diagram of the shape P_4 in a monoidal 2-category (for Yang–Baxter equations we have the hexagon which is P_3). This interpretation, however, assumes the associativity of the monoidal structure. In the general case, the 'right' diagram would be of the shape KP_4 and not P_4. For still higher-dimensional generalizations—simplex equations [22]—the natural approach seems to be to consider diagrams of the shape KP_{n+2} in n-categories by introducing on

KP_{n+2} the structure of a composable pasting scheme in the sense of Johnson [12].

In general, it is usually rather difficult to prove that some combinatorially defined CW-complex is a ball (cf., e.g., [30]) unless one uses convex realization. To prove that our KP_n is a ball, we use the connection with the moduli space $\overline{M_{0,n+1}}$ of stable punctured curves [5, 17]. In [6] Deligne used collections of sheaves on these spaces to give an interpretation of the braided monoidal structure in a category. The relation of $\overline{M_{0,n+1}}$ to associahedra and permutohedra was discussed, though vaguely, in several physical papers [3, 24]. The most precise formulation, to our opinion, is the following:

The set of real *points* $\overline{M_{0,n+1}}(\mathbb{R})$ *of* $\overline{M_{0,n+1}}$ *is glued from* $(1/2)n!$ *copies of the associahedra* K_n *by identifying some faces. These copies of* K_n *are in natural bijection with pairs of opposite vertices of the permutohedron* P_n.

Thus $\overline{M_{0,n+1}}(\mathbb{R})$ is temptingly close to the permuto-associahedron. The factor $(1/2)$ in the above formulation comes from the fact that $\overline{M_{0,n+1}}(\mathbb{R})$ is an iterated blow-up of a real projective space $\mathbb{R}P^{n-2}$ and not a sphere. There is an obvious double covering of $\overline{M_{0,n+1}}(\mathbb{R})$ (which we denote $\tilde{S}^{n-2}$, see Section 4). But it is still far from being a sphere since blow-ups glue in new projective spaces and not balls. So the relation between the two objects is not quite trivial.

We give in Section 4 an interpretation of the permutoassociahedron KP_n in terms of the geometry of $\overline{M_{0,n+1}}(\mathbb{R})$. Then, by using a representation of $\overline{M_{0,n+1}}$ as an iterated blow-up of P^{n-2} (given in [14, 16]), we prove in Section 5 that KP_n is indeed a ball.

A seemingly different way of application of Mac Lane's coherence theorem in quantum group theory was developed by Drinfel'd [8] in his study of the Knizhnik–Zamolodchikov (KZ) equation

$$\mathrm{d}f = \Big(\sum_{i<j} \Omega_{ij}\, \mathrm{d}\log(x_i - x_j)\Big) \cdot f$$

on a vector-function f on $\mathbb{R}^n$ (see [5]). The hyperplanes (mirrors) $\{x_i = x_j\}$ which are singular for this equation dissect $\mathbb{R}^n$ into $n!$ simplicial cones (Weyl chambers) corresponding to various orderings of the x_i. The CW-complex dual to the resulting decomposition of $\mathbb{R}^n$ is exactly the permutohedron P_n. The decisive step in Drinfel'd's analysis is the study of solutions of KZ for $n = 3,4$ with prescribed asymptotics in certain 'zones'. In general, each Weyl chamber contains as many zones as there are bracketings of n letters. The problem of constructing the permuto-associahedron would be trivial if one could just decompose each Weyl chamber into a union of these zones (then it would be sufficient to take the dual cell complex). However, the zones are not quite well defined as sets in $\mathbb{R}^n$ since the defining conditions use the sign $\ll$ (much less than) which does not have a precise meaning for real numbers. Clearly the sign $\ll$ can be made precise over a *non-archimedean* field. This is, of course, satisfactory for asymptotic studies (when x_i can be thought of as depending on a parameter t, i.e., belonging to a non-archimedean field $\mathbb{R}((t))$). But as subsets of $\mathbb{R}^n$ the zones are defined only 'up to shrinking'. We give in Section 3 a simple treatment of zones by means of the Bruhat–Tits tree and in Section 4 a comparison of the spatial position of zones and the vertices of permutoassociahedron KP_n. The relevance of moduli spaces to Drinfel'd's ones was pointed out in [6].

1. Associahedron and permutohedron

Let $A_1, \ldots, A_n$ be formal symbols. By a partial bracketing of $A_1, \ldots, A_n$ we shall mean an insertion into the formal product $A_1 \ldots A_n$ of several pairs of brackets in such a way that the result is meaningful. Thus $A_1(A_2A_3A_4)A_5$ is a partial bracketing and $A_1()A_2)(A_3A_4A_5$ is not. The empty bracketing $A_1 \ldots A_n$ is also allowed. Denote the set of all such partial bracketings by $\mathcal{B}_n$. The set $\mathcal{B}_n$ has a natural partial order: a bracketing β is said to be greater than another bracketing β' if β' may be obtained from β by adding new pairs of brackets. Thus the empty bracketing is the maximal element of $\mathcal{B}_n$ and minimal elements are precisely *complete* bracketings, which may be evaluated in a non-associative algebra. There is a well-known correspondence between complete bracketings and triangulations of a convex $(n+1)$-gon [4, 29]. Namely, consider such a polygon P in the plane and order vertices in the cyclic order: $x_0, \ldots, x_n$. Write the letter A_i on the edge $[x_{i-1}, x_i]$ of P. Let T be any triangulation of P. Let us orient all the edges $[x_i, x_j]$, $i<j$ of T to be directed from x_i to x_j. Then we can write at each edge $[x_i, x_j]$ of T a bracketing of $A_{i+1}, \ldots, A_j$ by applying inductively the following rule: If $i<j<k$ and we have already written bracketings U,V on $[x_i, x_j]$ and $[x_j, x_k]$ then the bracketing on $[x_i, k_k]$ is their product (in which U and V are taken in brackets). See Fig. 1 for an example.

In a similar way, partial bracketings are in bijection with decompositions of P into several polygons (not necessarily triangles).

Using the correspondence with triangulations, the associahedron can be constructed as follows (cf. [10]). Associate to any triangulation T of P the following vector $\phi(T)$ in the space $\mathbb{R}^{n+1}$ with coordinates $t_0, \ldots, t_n$,

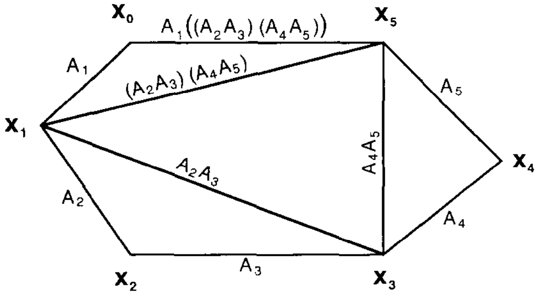


Fig. 1.

$$\phi(T)_i = \sum_{x_i \in \mathrm{Vert}(\tau)} \mathrm{Area}(\tau)\,,$$

where the summation is taken over all the triangles τ of T for which the point $x_i \in P$ is a vertex. By definition, K_n is the convex hull of all vectors $\phi(T)$. Thus there is a freedom in constructing a convex model for K_n; we can choose the shape of P. General results of [9] on triangulations imply the following fact.

Proposition 1.1. *The poset of faces of the polytope K_n is naturally isomorphic to the poset of all polygonal decompositions of P, i.e., to the poset $\mathcal{B}_n$ of bracketings of $A_1, \ldots, A_n$.* □

Example 1.2. The first three associahedra are given by the pictures in Fig. 2.

Let us now turn to the permutohedron.

Definition 1.3. The $(n-1)$*-dimensional permutohedron* P_n is the convex hull of $n!$ points $(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$, where σ runs over all the permutations of $\{1, \ldots, n\}$.

It is clear from this definition that P_n lies in the hyperplane

$$\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum x_i = n(n-1)/2\right\}$$

and its dimension equals $(n-1)$.

Let S_n be the symmetric group of all permutations of $\{1, \ldots, n\}$. For a permutation $\sigma = (\sigma(1), \ldots, \sigma(n)) \in S_n$ we shall denote by $[\sigma]$ the point $(\sigma^{-1}(1), \ldots, \sigma^{-1}(n)) \in P_n$. The advantage of this notation is seen from the following examples.

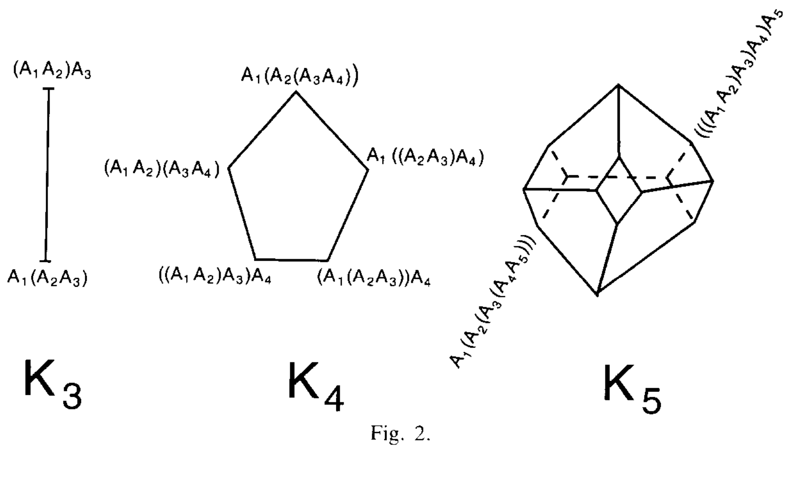
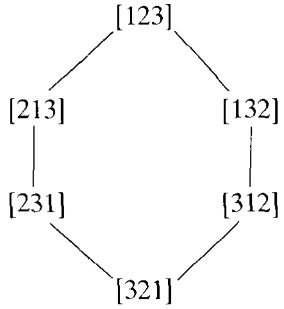


Fig. 2.

Example 1.4. The 2-dimensional permutohedron P_3 is the hexagon



and the permutohedron P_4 is the polytope in Fig. 3.

This polytope plays a crucial role in the interpretations of tetrahedra equations given in [15] and [19].

We see that two vertices of a permutohedron are connected by an edge if and only if the corresponding permutations are obtained from each other by interchanging two numbers standing in consecutive positions. If we had introduced the straightforward notation for the vertices, i.e. denoted by (σ) the vertex $(\sigma(1), \dots, \sigma(n))$ then the fact that two vertices are connected by an edge would mean that the two permutations are obtained from each other by interchanging some two consecutive *numbers*, e.g. 5 and 6, regardless of the position where these numbers stand in the permutation.

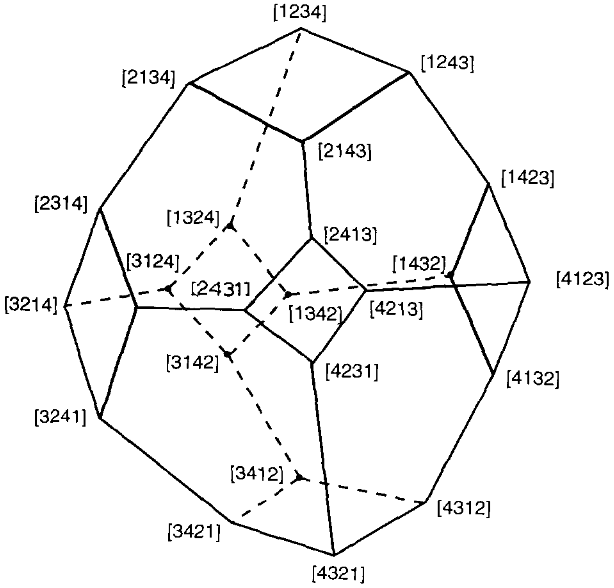


Fig. 3.

It is possible to describe all the faces of a general permutohedron P_n. To do this, we introduce the following terminology. By a *filtration* of $\{1, \ldots, n\}$ of length k we shall mean an increasing chain $I. = (\emptyset \subset I_1 \subset \cdots \subset I_k \subset \{1, \ldots, n\})$ of proper subsets of $\{1, \ldots, n\}$ where all the inclusions are strict. We say that a filtration $I.$ is a *refinement* of a filtration $J.$ if each J_p occurs among the I_q. The relation '$I.$ is a refinement of $J.$' defines a partial order on the set of all filtrations. Minimal elements with respect to this order are the filtrations which at each step add only one element. Such filtrations are in one-to-one correspondence with permutations $\sigma \in S_n$; we associate to σ the filtration

$$\{\sigma(1)\} \subset \{\sigma(1), \sigma(2)\} \subset \cdots \subset \{\sigma(1), \ldots, \sigma(n)\} .$$

For any filtration $I.$, let $R(I.)$ be the set of permutations whose associated filtrations are refinements of $I.$.

Theorem 1.5. (a) *All the points* $[\sigma]$, $\sigma \in S_n$, *are indeed vertices of the permutohedron* P_n.

(b) *Faces of* P_n *of codimension k are in bijection with filtrations of* $\{1, \ldots, n\}$ *of length k. More precisely, the face* $\Gamma(I.)$ *corresponding to the filtration* $I.$ *has as vertices the points* $[\sigma]$, $\sigma \in R(I.)$.

(c) *If* $I.$ *and* $J.$ *are two filtrations of* $\{1, \ldots, n\}$, *then* $\Gamma(I.) \subset \Gamma(J.)$ *if and only if* $I.$ *is a refinement of* $J.$.

(d) *Let* $I. = (I_1 \subset \cdots \subset I_k \subset \{1, \ldots, n\})$ *be a filtration of* $\{1, \ldots, n\}$ *of length k. Then, as a convex polytope,* $\Gamma(I.)$ *is affinely isomorphic to the product of permutohedra*

$$P_{|I_1|} \times P_{|I_2 - I_1|} \times \cdots \times P_{|\{1,\ldots,n\} - I_k|} . \qquad \square$$

Corollary 1.6. (a) *Two vertices* $[\sigma]$ *and* $[\tau]$ *of* P_n *are connected by an edge if and only if* σ *can be obtained from* τ *by interchanging two symbols standing in consecutive positions. Thus each vertex of* P_n *lies exactly on* $(n-1)$ *edges.*

(b) *There are exactly* $2^n - 2$ *faces of* P_n *of codimension* 1; *these faces correspond to proper subsets of* $\{1, \ldots, n\}$ (*filtrations of length* 1). $\square$

The proofs of Theorem 1.5 and Corollary 1.6 can be found in [23] or deduced from results on a more general class of polytopes introduced by Gelfand and Serganova [11] and called general hypersimplices.

The cell complex formed by the faces of the permutohedron admits several other useful descriptions. Let us recall one of them [1] representing P_n as a kind of 'loop space' of the n-dimensional cube.

Consider the unit cube $I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon 0 \le x_i \le 1\}$. It has 2^n vertices which are in bijection with subsets of $\{1, \ldots, n\}$. More precisely, let e_i, $i = 1, \ldots, n$, be the ith standard basis vector of $\mathbb{R}^n$. If J is a subset of $\{1, \ldots, n\}$, we will denote by e_J the vertex $\sum_{j\in J} e_j$ of I^n. The subsets of $\{1, \ldots, n\}$ are naturally ordered by inclusion and hence we have a partial order on vertices of I^n. This ordering can also be obtained by means of the linear function $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ on I^n. It is immediate to see that each face $\Gamma \subset I^n$ has a unique minimal vertex $\alpha(\Gamma)$ and a unique maximal vertex $\omega(\Gamma)$ with respect to the said order. Call a *cellular string* in I^n a collection $(\Gamma_1, \ldots, \Gamma_k)$ of faces such that

$$\alpha(\Gamma_1) = \alpha(I^n) = (0, \ldots, 0), \qquad \omega(\Gamma_k) = \omega(I^n) = (1, \ldots, 1)$$

and for $i = 2, \ldots, k$ we have $\alpha(\Gamma_i) = \omega(\Gamma_{i-1})$. We identify a cellular string with the subcomplex in the cube formed by its cells. The inclusion of subcomplexes defines a partial order relation on the set of all cellular strings.

Proposition 1.7. *The poset of all cellular strings in the cube I^n is isomorphic to the poset of faces of the permutohedron P_n.*

Proof. Each face $\Gamma \subset I^n$ has a 'direction' which is the coordinate subspace in $\mathbb{R}^n$ parallel to Γ. In other words, the direction is a subset $D(\Gamma)$ in $\{1, \ldots, n\}$ of cardinality $\dim(\Gamma)$. It is almost immediate to see that the correspondence

$$(\Gamma_1, \ldots, \Gamma_k) \mapsto (D(\Gamma_1) \subset D(\Gamma_1) \cup D(\Gamma_2) \subset \cdots \subset D(\Gamma_1) \cup \cdots \cup D(T_k))$$

establishes a bijection between the set of cellular strings and the set of filtrations of $\{1, \ldots, n\}$. □

For example, a minimal element in the poset of cellular strings is just a monotone edge path in the cube starting at the vertex $(0, \ldots, 0)$ and ending at $(1, \ldots, 1)$. Considering the directions of edges of this path, we obtain a permutation of $\{1, \ldots, n\}$.

The above description will be the starting point for the definition of permutoassociahedron in Section 2. Another (almost trivial) description is the following:

Proposition 1.8. *Consider the CW-decomposition of the space $\mathbb{R}^n$ into chambers cut by hyperplanes $\{x_i = x_j\}$. The poset of cells of this decomposition is anti-isomorphic to the poset of faces of the permutohedron P_n (in other words, these decompositions are dual to each other).* □

2. The permutoassociahedron

We start by defining the poset $\mathcal{F}_n$ which will be the poset of faces of the permutoassociahedron KP_n. As in Proposition 1.6 consider the cube I^n and cellular strings in it.

Definition 2.1. A *bracketed string in* I^n is a pair $S = (\Sigma, \beta)$, where $\Sigma = (\Gamma_1, \ldots, \Gamma_k)$ is a cellular string in I^n and β is a partial bracketing in the formal product $\Gamma_1 \ldots \Gamma_k$.

Denote the set of all bracketed strings in I^n by $\mathscr{F}_n$. We proceed to define a partial order on $\mathscr{F}_n$. Let $\mathscr{P}_n$ be the poset of faces of the permutohedron P_n (i.e. of unbracketed cellular strings in I^n).

Suppose that $\Sigma = (\Gamma_1, \ldots, \Gamma_k)$ and $\Sigma' = (\Delta_1, \ldots, \Delta_l)$ are two elements of $\mathscr{P}_n$ such that $\Sigma \leq \Sigma'$. This means, by definition, that for each $i \in [1, l]$ there is a segment $[a_i, b_i] \subset [1, k]$ such that $(\Gamma_{a_i}, \Gamma_{a_i+1}, \ldots, \Gamma_{b_i})$ is a cellular string in the face Δ_i. In this situation let β be a partial bracketing of $\Delta_1 \ldots \Delta_l$. By replacing in this bracketing each Δ_i by the (unbracketed) product $\Gamma_{a_i} \ldots \Gamma_{b_i}$, we obtain a bracketing in $\Gamma_1 \ldots \Gamma_k$ which we call the bracketing induced by β and denote $i_{\Sigma,\Sigma'}(\beta)$.

Definition 2.2. Let $S = (\Sigma, \beta)$ and $S' = (\Sigma', \gamma)$ be two bracketed strings. We say that $S \leq S'$ if $\Sigma \leq \Sigma'$ in P_n and $\beta \leq i_{\Sigma,\Sigma'}(\gamma)$ in $\mathscr{B}_k$, where k is the number of faces in the cellular string Σ.

For any poset $\mathscr{X}$ we shall denote by $|\mathscr{X}|$ its nerve, see [1, 2, 27, 30]. By definition, this is a simplicial complex whose p-simplices correspond to chains $x_0 < \cdots < x_p$ of strictly increasing elements in $\mathscr{X}$. If $x \in \mathscr{X}$ is any element of the poset, then we shall denote by $\leq(x)$ the subposet $\{y \in \mathscr{X}: y \leq x\}$.

Definition 2.3. The permutoassociahedron KP_n is the nerve $|\mathscr{F}_n|$ of the poset $\mathscr{F}_n$. For any element $S \in \mathscr{F}_n$ we denote by $[S]$ the nerve of the subposet $\leq(S)$.

We shall show below that each $[S]$ is in fact a cell and thus the $[S]$ provide a CW-decomposition of KP_n. Before that let us consider two examples in which this CW-decomposition can be seen explicitly.

Examples 2.4. (a) The third permutoassociahedron KP_3 considered with its CW-decomposition given by the $[S]$ is the dodecagon (Fig. 4) which replaces the familiar Yang–Baxter hexagon in weak monoidal categories. This dodecagon first appeared in [21].

(b) The 'polytope' KP_4 contains 120 vertices and is difficult to draw in detail. Here are simple directions how to do this. Start from the permutohedron P_4. Its faces are squares and hexagons. Inside each square mark one point near each of the four vertices. Inside each hexagon mark two points near each vertex. Each vertex of P_4 is contained in three faces: one square and two hexagons. Thus there will be five marked points near each vertex. Join them by edges to form a pentagon and then join these pentagons to each other as shown in Fig. 5.

In this way each hexagon will be replaced by a dodecagon, each square by a

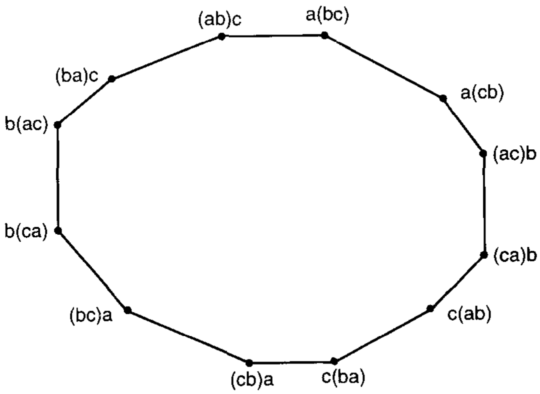


Fig. 4.

smaller square, each vertex will be blown up to a pentagon and each edge will be doubled to give a rectangle.

The following theorem is the main result of this paper.

Theorem 2.5. *The simplicial complex KP_n is homeomorphic to an $(n-1)$-dimensional ball. Each subcomplex $[S]$ is also a ball and these subcomplexes form a CW-decomposition of KP_n.*

The proof of this theorem will be given in Section 5.

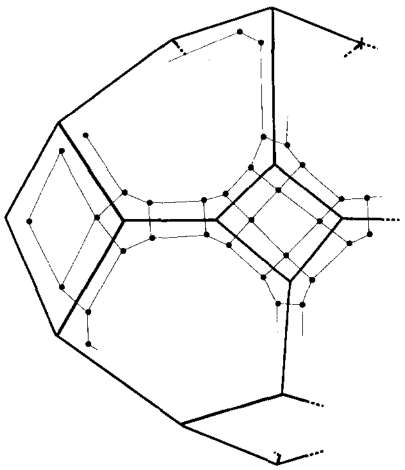


Fig. 5.

3. Drinfel'd's asymptotic zones and the Bruhat–Tits tree

In this section we give a detailed discussion of Drinfel'd's zones associated to bracketings, supplying details implicit in [8].

Denote by $\mathbb{R}^{n-1}$ the vector space of n-tuples $(x_1, \ldots, x_n)$ of real numbers with zero sum. Denote by $\mathscr{A}_{n-1}$ the configuration of hyperplanes in $\mathbb{R}^{n-1}$ formed by the hyperplanes $\{x_i = x_j\}$. For any permutation $\sigma \in S_n$ denote by K_σ the component of $\mathbb{R}^{n-1} - \mathscr{A}_{n-1}$ given by inequalities $x_{\sigma(1)} < \cdots < x_{\sigma(n)}$. This is an open simplicial cone.

Definition 3.1. Let $A_1, \ldots, A_n$ be formal symbols, $\sigma \in S_n$ a permutation and β a complete bracketing on the permuted product $A_{\sigma(1)} \ldots A_{\sigma(n)}$. Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a germ at $t = 0$ of a real analytic map $\mathbb{R} \to \mathbb{R}^{n-1}$. We shall say that $x(t)$ belongs to the asymptotic zone $K_{\sigma,\beta}$ if the following conditions hold:

(1) For small $t > 0$ we have $x_{\sigma(1)}(t) < \cdots < x_{\sigma(n)}(t)$.

(2) Let i, j, k be three pair of indices such that in the bracketing β there is a pair of brackets containing $A_{\sigma(i)}$ and $A_{\sigma(j)}$ but not containing $A_{\sigma(k)}$. Then $\mathrm{ord}_t(x_{\sigma(i)}(t) - x_{\sigma(j)}(t)) > \mathrm{ord}_t(x_{\sigma(i)}(t) - x_{\sigma(k)}(t))$.

Remark 3.2. We can try to visualize the zone $K_{\sigma,\beta}$ as a subset of the simplicial cone K_σ by replacing the inequality on orders in condition (2) above with the inequality $|x_{\sigma(i)} - x_{\sigma(j)}| \ll |x_{\sigma(i)} - x_{\sigma(k)}|$ on coordinates of a point in K_σ. Though the sign $\ll$ ('much less than') does not have a precise meaning when applied to fixed real numbers, it may help to visualize the zones $K_{\sigma,\beta}$. For example, twelve zones in $(\mathbb{R}^2, \mathscr{A}_2)$ can be depicted as in Fig. 6.

The meaning of condition (2) of Definition 3.1 can be clarified by using the

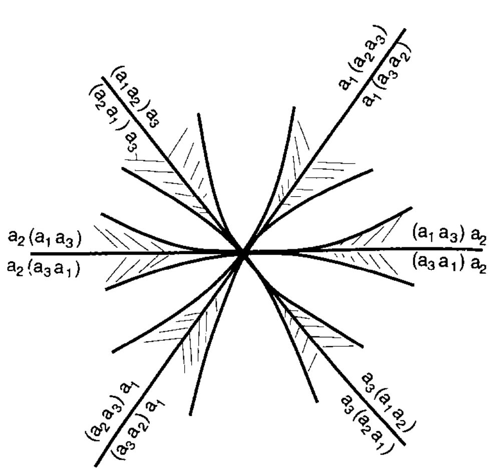


Fig. 6.

well-known correspondence between bracketings and rooted trees [2, 4, 27]. Let us recall this correspondence which goes back to A. Cayley.

Let β be a (not necessarily complete) bracketing in the formal product $A_1 \dots A_n$. A segment $s = (A_i \dots A_j)$ is called *admissible* with respect to β if it contains with every right bracket the left bracket corresponding to it and similarly for left brackets. For example, each A_i as well as the whole product $A_1 \dots A_n$ are admissible. The set of admissible segments is ordered by inclusion and with this order the A_i are minimal elements and $A_1 \dots A_n$ is the maximal one.

Definition 3.3. Let β be a (partial) bracketing in $A_1 \dots A_n$. Its tree $\mathcal{T}(\beta)$ has as vertices all the admissible segments in β and another new vertex A_0. Two admissible segments s and s' are joined by an edge if $s \subset s'$ and there are no other admissible segments between s and s'. The new vertex A_0 is joined by a new edge with the vertex $A_1 \dots A_n$.

The construction of the tree $\mathcal{T}(\beta)$ is clear from Fig. 7.

It is immediate to see that $\mathcal{T}(\beta)$ is just the dual graph to the polygonal decomposition of an $(n+1)$-gon corresponding to β (see Section 1).

In any tree, we call the *valency* of a vertex v the number of edges containing v. Vertices of valency 1 are called *endpoints*.

The following proposition is elementary and well-known [4].

Proposition 3.4. *The correspondence $\beta \mapsto \mathcal{T}(\beta)$ induces a bijection between $\mathcal{B}_n$ (the set of all partial bracketings in $A_1 \dots A_n$) and the set of isomorphism classes of plane trees with endpoints $A_0, A_1, \dots, A_n$ and no other vertices of valency ≤ 2. In particular, complete bracketings correspond to trees in which any vertex other than $A_0, \dots, A_n$, has valency 3.* □

The connection of the above construction with Definition 3.1 is provided by using a special class of infinite trees called Bruhat–Tits trees [26]. We shall use here the following version of this construction.

$(A_1(A_2A_3))A_4$

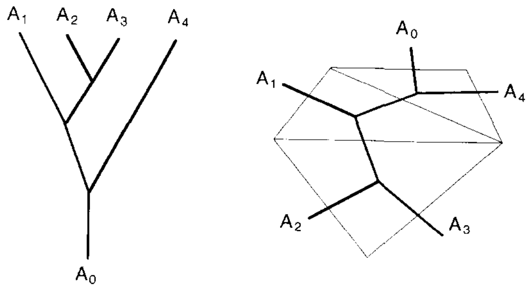


Fig. 7.

Definition 3.5. Let k be a field and $k[[t]]$ be the ring of formal power series over k. The Bruhat–Tits tree of $k[[t]]$, denoted $\mathrm{BT}(k[[t]])$, has the following vertices. First, there are vertices corresponding to pairs (f, n) where $n = -1, 0, 1, \ldots$ and $f \in k[t]$ is a polynomial of degree $\leq n$ (we assume that the only polynomial of degree (-1) is 0). In addition, there is one vertex denoted A_0. Two vertices (f, n) and (g, m) are joined by an edge if $|n - m| - 1$ (say, $m = n + 1$) and $f \equiv g \pmod{t^{n+1}}$. The vertex A_0 is joined by a new edge to the vertex $(-1, 0)$.

The structure of $\mathrm{BT}(k[[t]])$ is depicted in Fig. 8.

Let $v = (f(x) = a_0 + \cdots + a_n t^n, n)$ be any vertex of $\mathrm{BT}(k[[t]])$ such that $n \geq 0$. Edges containing v are easy to describe. Namely, there is the edge joining v with $(a_0 + \cdots + a_{n-1}t^{n-1}, n-1)$ (it goes upwards in Fig. 8) and there are edges $(a_0 + \cdots + a_n t^n + \lambda t^{n+1}, n+1)$ for any $\lambda \in k$. In Fig. 8 these edges are directed downwards from v. Thus downward edges from v are parametrized by k. It is convenient to associate to the unique upward edge from v the infinite point ∞ on the projective line $P^1(k) = k \cup \{\infty\}$. Therefore the set of all edges containing any given vertex of $\mathrm{BT}(k[[t]])$ other than A_0 is naturally identified with $P^1(k)$.

Very important for us will be the following easy remark.

Proposition 3.5. *The set $k[[t]]$ is identified with the set of infinite edge paths in $\mathrm{BT}(k[[t]])$ starting at the top vertex A_0 and not passing any point twice.* □

Remark 3.6. Usually (cf. [28]) one calls the Bruhat–Tits tree a bigger tree whose vertices are cosets from $\mathrm{SL}_2(k((t)))/\mathrm{SL}_2(k[[t]])$ and whose 'ends' form the projective line $P^1(k((t)))$. Our tree is clearly a subtree of this bigger tree.

Let now $f_1(t), \ldots, f_n(t)$ be n formal power series from $k[[t]]$. Let

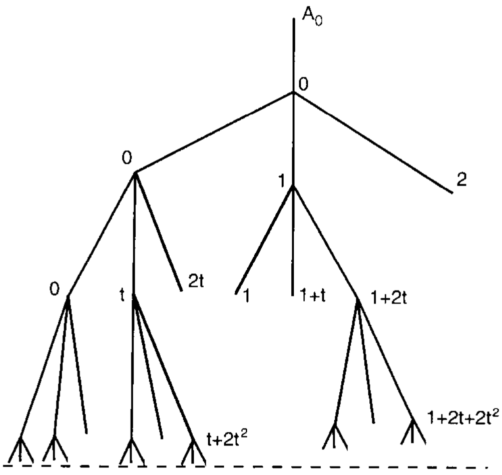
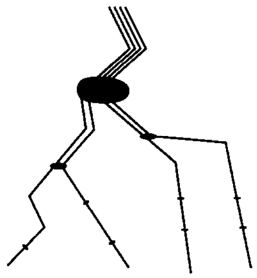


Fig. 8.



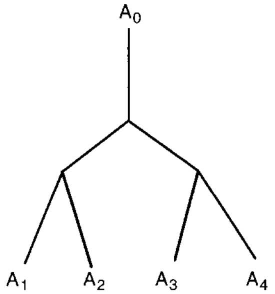


Fig. 9.

$\Gamma(f_1, \ldots, f_n)$ be the union of edge paths in $\mathrm{BT}(k[[t]])$ corresponding to f_i. Topologically, this figure is a tree. We define a combinatorial tree $\mathcal{T}(f_1, \ldots, f_n)$ associated to this figure, as follows. By definition, vertices of $\mathcal{T}(f_1, \ldots, f_n)$ are of two kinds:

(1) The 'ends' of $\Gamma(f_1, \ldots, f_n)$ i.e. the top vertex A_0 and endpoints A_i, $i = 1, \ldots, n$ corresponding to paths associated to f_i.

(2) All those vertices of $\Gamma(f_1, \ldots, f_n)$ which have valency ≥ 3. Edges in $\mathcal{T}(f_1, \ldots, f_n)$ are induced by edge paths in $\Gamma(f_1, \ldots, f_n)$. In other words, we straighten topologically trivial stretches in $\Gamma(f_1, \ldots, f_n)$ (see Fig. 9).

Proposition 3.7. *A real analytic germ of a map $(x_1(t), \ldots, x_n(t)) : \mathbb{R} \to \mathbb{R}^n$ belongs to the asymptotic zone $K_{\sigma,\beta}$ (Definition 3.1) if and only if two conditions hold:*

(1) *For small $t > 0$ we have $x_{\sigma(1)}(t) < \cdots < x_{\sigma(n)}(t)$.*

(2) *The tree $\mathcal{T}(x_{\sigma(1)}(t), \ldots, x_{\sigma(n)}(t))$ is isomorphic to the tree $\mathcal{T}(\beta)$ (Definition 3.3).*

Proof. The proof is straightforward. □

4. Relation to the moduli space of stable $(n + 1)$-pointed curves

A better understanding of Drinfel'd's zones may be obtained by considering the moduli space $\overline{M_{0,n+1}}$ of stable pointed curves of genus 0 introduced by A. Grothendieck and F. Knudsen, see [5, 17]. These spaces were used by Deligne in [6] to give an interpretation of a braided monoidal structure in a category in terms of natural systems of sheaves on $\overline{M_{0,n+1}}$, $n \geq 0$. In what follows we shall use these ideas of Deligne to give a conceptual interpretation of zones and another construction of the permutoassociahedron. The main difference between our approach and that of [6] is that we consider real points.

Let us recall from [17] the definitions related to $\overline{M_{0,n+1}}$.

Definition 4.1. A *stable* $(n+1)$*-pointed curve of genus* 0 *over a field* k is a (possibly reducible) curve C over k together with $(n+1)$ distinct smooth points $x_0, \dots, x_n \in C$ defined over k, satisfying the following conditions:

(1) C has only ordinary double points and every irreducible component of C is isomorphic to the projective line P_k^1. The points of intersections of components are also k-rational.

(2) The graph of components of C is a tree.

(3) On each component of C there are at least three points which are either marked or double.

By definition, a double point of C cannot be marked (since marked points are smooth). Points of C which are either marked or double will be called *special*.

We shall use the following 'dual' point of view on the graph of components of a curve.

Definition 4.2. Let $(C, x_0, \dots, x_n)$ be a stable $(n+1)$-pointed curve of genus 0. Its tree $\mathcal{T}(C, x_0, \dots, x_n)$ has the following vertices:

(1) Endpoints (1-valent vertices) $A_0, \dots, A_n$ corresponding to $x_0, \dots, x_n$.

(2) Vertices corresponding to all the components of C.

Two vertices of type (2) are joined by an edge if the corresponding components intersect. An endpoint A_i is joined by an edge to the vertex of type (2) corresponding to the unique component containing the point x_i.

Definition 4.2 is illustrated in Fig. 10.

As shown in [17] there exists the moduli space $\overline{M_{0,n+1}}$ of stable $(n+1)$-pointed curves of genus 0 which is a smooth projective variety of dimension $(n-2)$ defined over the rational numbers (and even over $\mathbb{Z}$). For the time being we consider it as a complex variety.

The variety $\overline{M_{0,n+1}}$ contains an open subset $M_{0,n+1}$ consisting of $(C, x_0, \dots, x_n)$

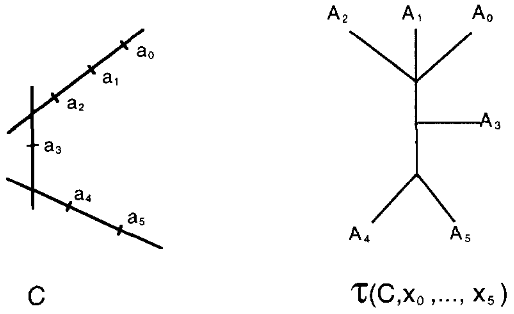


Fig. 10.

where $C = P^1$ and x_i are distinct points on it. Thus

$$M_{0,n+1} = ((P^1)^{n+1} - \bigcup \{x_i = x_j\})/\mathrm{GL}_2 .$$

Lemma 4.3. *The variety $M_{0,n+1}$ is isomorphic to an open subset in*

$$P^{n-2} = P\left(\left\{(a_1, \ldots, a_n) \in \mathbb{C}^n : \sum a_i = 0\right\}\right),$$

namely the complement of the hyperplane configuration $P(\mathcal{A}_{n,\mathbb{C}})$ consisting of $\binom{n}{2}$ hyperplanes $\{a_i = a_j\}$.

Proof. Given a collection $(x_0, \ldots, x_n)$ of distinct points on P^1, there is a unique projective transformation taking x_0, x_1, x_2 to $\infty, 0, 1$ respectively. Thus $M_{0,n+1} = \{(x_3, \ldots, x_n) \in \mathbb{C}^{n-2}: x_i \neq x_j,\ x_i \neq 0,1\}$. By compactifying $\mathbb{C}^{n-1}$ to P^{N-2}, we represent $M_{0,n+1}$ as the complement of the configuration consisting of: the infinite hyperplane, the $\binom{n-2}{2}$ hyperplanes $x_i = x_j$, $i, j = 3, \ldots, n$ and $2n - 4$ hyperplanes $x_i = 0$, $x_i = 1$. This configuration is easily seen to be projectively isomorphic to $P(\mathcal{A}_{n,\mathbb{C}})$. $\square$

In fact, $\overline{M_{0,n+1}}$ can be represented as an iterated blow-up of P^{n-2} along subspaces which are intersections of hyperplanes from the configuration $P(\mathcal{A}_{n,\mathbb{C}})$, see [14]. The regular birational morphism $p : \overline{M_{0,n+1}} \to P^{n-2}$ is constructed as follows. Let $(C, x_0, \ldots, x_n) \in \overline{M_{0,n+1}}$ and let $C_0 \subset C$ be the component containing x_0. Then $C_0 - \{x_0\}$ is isomorphic to the affine line A^1. For each point $x_j \neq x_0$, let $y_j \in C_0 - \{x_0\}$ be the unique point of C_0 such that either $y_j = x_j$ (and so $x_j \in C_0$) or y_j is the unique intersection point with C_0 of a connected subcurve in C containing x_j but not C_0 (Fig. 11).

There is exactly one point $z = z(y_1, \ldots, y_n) \in C - \{x_0\} = A^1$ such that $\sum (z - y_i) = 0$. The point $p(C, x_0, \ldots, x_n) \in P^{n-2}$ is, by definition, the point with homogeneous coordinates $(z - y_1 : \cdots : z - y_n)$.

Example 4.4. The space $\overline{M_{0,5}}$ is the blow-up of P^2 in four points $(P_1, \ldots, P_4)$. The configuration $P(\mathcal{A}_{4,\mathbb{C}})$ consists of six lines (P_i, P_j) joining these points and the P_i are the triple points of this configuration.

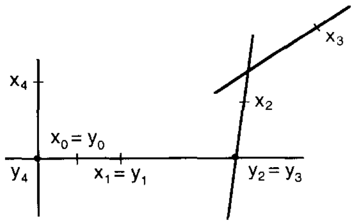


Fig. 11.

Remarks 4.5. (a) A more conceptual construction of the projection p is as follows [13]. Let $\mathscr{L}_0$ be the line bundle on $\overline{M_{0,n+1}}$ whose fiber at $(C, x_0, \ldots, x_n)$ is $T^*_{x_0}C$, the cotangent space to C at x_0. It was shown in [13] that $H^0(\overline{M_{0,n+1}}, \mathscr{L})$ has dimension $n-1$ and the corresponding rational map to $P((H^0)^*)$ is regular birational. This is precisely p. In [14] an explicit decomposition of p in blow-ups was given. A representation of $\overline{M_{0,n+1}}$ as a blow-up of $(P^1)^{n-2}$ was given by Keel [16].

(b) The compactness of $\overline{M_{0,n+1}}$ can be informally explained as follows. When two marked points x_i,x_j on a stable curve try to coincide (which is 'prohibited') the result is that the curve acquires a new component glued to the old curve at the intended point of coincidence and the two points in question land in this component. The precise position of them on this component is inessential since this new component contains only three special points: x_i, x_j and the point of intersection with the old part (note that all triples of distinct points on P^1 are projectively equivalent!).

The relevance of $\overline{M_{0,n+1}}$ to associahedra stems from the fact that both are stratified with strata labelled by trees. More precisely, we give the following definition.

Definition 4.6. Let $\mathscr{T}$ be a tree bounding $(n+1)$ endpoints $A_0, \ldots, A_n$. We denote by $\overline{M_{0,n+1}}(\mathscr{T})$ the subvariety in $\overline{M_{0,n+1}}$ consisting of $(C, x_0, \ldots, x_n)$, whose tree (Definition 4.2) is isomorphic to $\mathscr{T}$. By a stratum in $\overline{M_{0,n+1}}$ we shall mean a subvariety of the form $\overline{M_{0,n+1}}(\mathscr{T})$.

The structure of the strata as algebraic varieties is as follows.

Proposition 4.7. *Let $\mathscr{T}$ be a tree bounding endpoints $A_0, \ldots, A_n$. For each vertex $v \in \mathscr{T}$ other than A_i let $e(v)$ be its valency (which is ≥ 3). We have an isomorphism*

$$\overline{M_{0,n+1}}(\mathscr{T}) \equiv \prod_{v \in \mathscr{T}} M_{0,e(v)} \, .$$

In particular, $\overline{M_{0,n+1}}(\mathscr{T})$ is a point if and only if all $e(v)$ are equal to 3.

Proof. The proof is immediate. $\square$

From now on, we shall be interested only in *real* points of $\overline{M_{0,n+1}}$. Correspondingly, we call strata in $\overline{M_{0,n+1}}(\mathbb{R})$ the sets $\overline{M_{0,n+1}}(\mathscr{T})(\mathbb{R})$ of real points of $\overline{M_{0,n+1}}(\mathscr{T})$. By a *cell* in $\overline{M_{0,n+1}}(\mathbb{R})$ we shall mean a connected component of some stratum.

Proposition 4.8. *Each cell in $\overline{M_{0,n+1}}(\mathbb{R})$ is a cell in the topological sense. All together they define a CW-decomposition of $\overline{M_{0,n+1}}(\mathbb{R})$. Each maximal cell is*

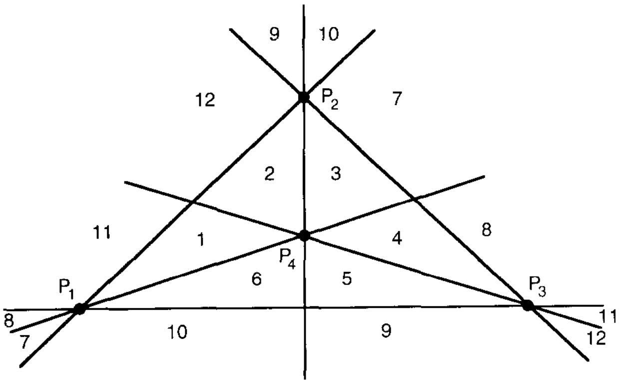


Fig. 12.

combinatorially equivalent to the associahedron K_n and the number of maximal cells is $(1/2)n!$.

Example 4.9. The space $\overline{M_{0,5}}(\mathbb{R})$ is the blow-up of $\mathbb{R}P^2$ in four points $P_1, \ldots, P_4$ (Example 4.4). Each of the twelve triangles cut out by six lines (P_i, P_j) (see Fig. 12) contains two of the points P_i. Thus after the blow-up each triangle will become a pentagon (i.e. the associahedron K_4).

Proof of Proposition 4.8. Consider the map $p : \overline{M_{0,n+1}}(\mathbb{R}) \to \mathbb{R}P^{n-2}$ constructed above. It is one-to-one outside the configuration $P(\mathscr{A}_n) \subset \mathbb{R}P^{N-2}$. The hyperplanes of this configuration cut $\mathbb{R}P^{N-2}$ into $(1/2)n!$ simplices, since opposite Weyl chambers K_σ and $K_{\sigma w_0}$ in $\mathbb{R}^{n-1}$ will have the same image under projectivisation (here w_0 is the permutation $(n, n-1, \ldots, 1)$). Hence maximal 'cells' in $\overline{M_{0,n+1}}(\mathbb{R})$, being preimages of these simplices $P(K_\sigma)$, are topological cells. For the 'cells' of smaller dimension, which are products of maximal cells in $M_{0,n_i}(\mathbb{R})$, the assertion follows.

Now we claim that after blow-ups provided by p, the $(n-2)$-dimensional simplices $P(K_\sigma) \subset P^{n-2}$ will become associahedra. But this follows from the fact that both are stratified into strata labelled by trees of the same type (Definitions 4.6 and 3.3). □

As we have seen, $\overline{M_{0,n+1}}(\mathbb{R})$ can be seen as an iterated blow-up of $\mathbb{R}P^{n-2}$ desingularizing the non-normal crossing divisor $P(\mathscr{A}_n)$. It will be more convenient for us to use a version of this space which 'desingularizes' in the same sense the unit sphere S^{n-2}. By using the approach of Deligne [6], we define the space $\tilde{S}^{n-2}$ as the moduli space of collections $(C, x_0, \ldots, x_n, \omega)$, where $(C, x_1, \ldots, x_n)$ is a stable $(n+1)$-pointed curve of genus 0 over $\mathbb{R}$ and ω is an orientation of the component of C (circle) which contains x_0. There is a canonical commutative diagram

$$(*)\qquad \begin{array}{ccc} \tilde{S}^{n-2} & \longrightarrow & \overline{M_{0,n+1}}(\mathbb{R}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S^{n-2} & \longrightarrow & \mathbb{R}P^{n-2} \end{array}$$

where horizontal arrows are two-sheeted coverings. We endow $\tilde{S}^{n-2}$ with the natural CW-structure induced from $\overline{M_{0,n+1}}(\mathbb{R})$. The sphere S^{n-2} will always be considered with the CW-structure given by the decomposition into $n!$ spherical simplices induced by Weyl clambers K_σ.

Proposition 4.8 implies the following fact about $\tilde{S}^{n-2}$:

Proposition 4.10. *$\tilde{S}^{n-2}$ is the union of $n!$ copies of the associahedron K_n. These copies are in bijection with vertices of the permutohedron P_n.* □

We shall note by $S^{n-2}(\sigma)$ the associahedron corresponding to $\sigma \in S_n$.

As we noted in the Introduction, $\tilde{S}^{n-2}$ is still far from being a sphere, since blow-ups glue in projective bundles. We are going now to give a construction of Drinfel'd's zones and the permuto-associahedron KP_n in terms of $\tilde{S}^{n-2}$.

Let $\pi : \mathbb{R}^{n-1} - \{0\} \to S^{n-2}$ be the canonical projection and $q : \tilde{S}^{n-2} \to S^{n-2}$ the vertical projection in $(*)$.

Theorem 4.11. *Let $x(t) = (x_1(y), \ldots, x_n(t))$, $\sum x_i(t) = 0$, be a real analytic germ of a map $\mathbb{R} \to \mathbb{R}^{n-1}$. Suppose that for a small positive t all $x_i(t)$ are distinct. Then $x(t)$ lies in the asymptotic zone $K_{\sigma,\beta}$ if and only if $q^{-1}(\pi(x(t)))$ lies inside the cell (associahedron) $\tilde{S}^{n-2}(\sigma) \subset \tilde{S}^{n-2}$ and $\lim_{t\to 0} q^{-1}(\pi(x(t)))$ is the vertex of the associahedron $\tilde{S}^{n-2}(\sigma)$ corresponding to the bracketing β.*

Proof. In Section 3 we have associated to any n-tuple $(f_1, \ldots, f_n)$ of formal power series over a field k, a tree $\mathcal{T}(f_1, \ldots, f_n)$ bounding $(n+1)$ endpoints; this was done by considering series as paths in the Bruhat–Tits tree. Taking into account Proposition 3.7, our theorem is a consequence of the following fact.

Proposition 4.12. *Let $x_1(t), \ldots, x_n(t)$ be analytic functions defined near 0. Consider the stable $(n+1)$-pointed curve which is the limit*

$$\lim_{t\to 0} (P^1, \infty, x_1(t), \ldots, x_n(t))$$

in $\overline{M_{0,n+1}}$. Then the tree of this curve (Definition 4.2) is isomorphic to $\mathcal{T}(x_1(t), \ldots, x_n(t))$. □

This proposition was proven in [13]. The proof of Theorem 4.11 is finished. □

Theorem 4.11 means that Drinfel'd's zones being lifted to $\tilde{S}^{n-2}$ become just

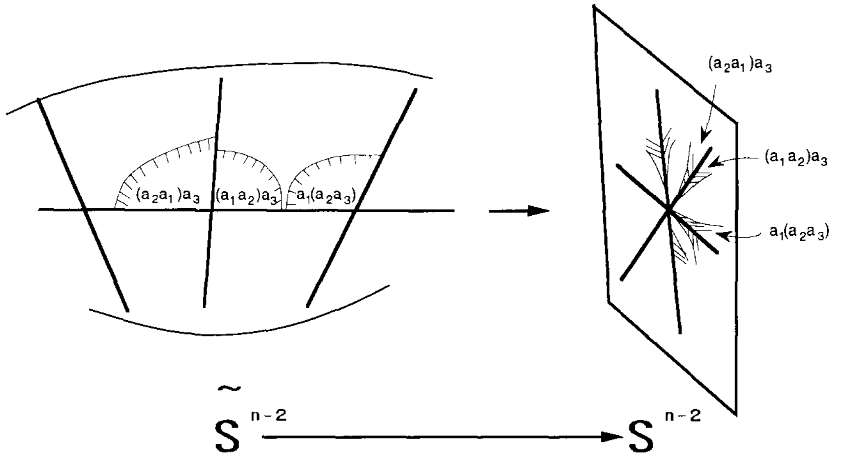


Fig. 13.

'corners' (i.e. regions near the vertices) of associahedra $\tilde{S}^{n-2}(\sigma)$. In Fig. 13 we try to illustrate this phenomenon.

In more formal terms, the set of zones (i.e., the set of vertices of the permutoassociahedron KP_n) is identified with the set of flags (b, B), where b is a 0-cell in $\tilde{S}^{n-2}$ and B is an $(n-2)$-cell containing it. Since $(n-2)$-cells in $\tilde{S}^{n-2}$ and in S^{n-2} are in bijection, we can code zones also by pairs (b, D), where b is as before and D is an $(n-2)$-cell (simplex) in S^{n-2} such that $b \in q^{-1}(D)$. This latter point of view can be generalized to give a description of the whole face lattice of KP_n in terms of the geometry of the projection $q: \tilde{S}^{n-2} \to S^{n-2}$.

Theorem 4.13. *The set $\mathscr{F}_n^0$ of proper faces of the permutoassociahedron KP_n is in natural bijection with the set of pairs (Γ, Σ) where Γ is a (closure of a) cell in S^{n-2}, Σ is a stratum in $\tilde{S}^{n-2}$ such that:*

(1) $q^{-1}(\Gamma)$ contains at least one connected component of Σ,

(2) $\dim(\Sigma) \geq \dim(q^{-1}(c)) - \dim(c)$.

Denoting by $c(\Gamma, \Sigma)$ the face of KP_n corresponding to (Γ, Σ), we have $c(\Gamma, \Sigma) \subset c(\Gamma', \Sigma')$ if and only if $\Gamma' \subset \Gamma$ and $\Sigma \subset \Sigma'$.

Proof. Cells in S^{n-2} are in bijection with proper faces of the permutohedron P_n or, in view of Proposition 1.7, cellular strings in the cube I^n. A choice of a stratum (i.e., of a tree, see Definition 4.6) as in the formulation of our theorem is equivalent to a choice of a partial bracketing on the formal product of cells from the cellular string. So our statement follows from Definition 2.1. □

5. Proof that KP_n is a ball

Here we prove Theorem 2.5.

1. We use induction on n and assume that for any $m < n$ the assertion about KP_m is proven. The maximal element in $\mathcal{F}_n$ is the cellular string given by I^n itself with trivial bracketing. Let $S = ((\Gamma_1, \ldots, \Gamma_k), \beta)$ be any other element of $\mathcal{F}_n$. It is immediate to see that the poset $\leq(S)$ (whose nerve is $[S]$) is isomorphic to

$$\mathcal{B}_k \times \mathcal{F}_{\dim \Gamma_1} \times \cdots \times \mathcal{F}_{\dim \Gamma_k} .$$

Therefore the fact that all these $[S]$ are cells follows from the inductive assumption. Let $\mathcal{F}_n^0 \subset \mathcal{F}_n$ be the subset obtained by deleting the maximal element (I^n), so that $|\mathcal{F}_n|$ is the cone over $|\mathcal{F}_n^0|$. It suffices therefore to prove that $|\mathcal{F}_n^0|$ is homeomorphic to an $(n-2)$-sphere.

2. We shall use the fact that the moduli space $\overline{M_{0,n+1}}$ can be obtained from the projective space P^{n-2} by iterated blow-up of linear subspaces which are faces (intersections of hyperplanes) of the configuration $P(\mathcal{A}_{n-1})$. This fact is proven in [14]. More precisely, there is a sequence of algebraic varieties and projections $X_i \xrightarrow{p_i} P^{n-2}$, $i = 0, \ldots, N$ such that:

(1) $X_0 = P^{n-2}$, $X_N = \overline{M_{0,n+1}}$;

(2) each X_i is obtained from X_{i-1} by blow-up of a subvariety which is the proper transform (under q_{i-1}) of some face of the configuration $P(\mathcal{A}_{n-1})$ in P^{n-2}.

Let $Y_i \xrightarrow{q_i} S^{n-2}$ be the obvious double cover of $X_i(\mathbb{R})$ with its projection to S^{n-2}.

This cover is obtained from S^{n-2} by iterated blowing up geodesic subspheres corresponding to faces of the configuration $\mathcal{A}_{n-1}$. We shall denote the configuration of 'big' spheres in S^{n-2} corresponding to hyperplanes from $\mathcal{A}_{n-1}$ by $S(\mathcal{A}_{n-1})$. We have the tower of natural maps $p_{ij} : Y_i \to Y_j$ where $i > j$. If Z is any hypersurface in Y_j and $i > j$ then by *strict transform* of Z we shall mean the closure of $p_{ij}^{-1}(Z_0)$ where $Z_0 \subset Z$ is the open set consisting of points over which p_{ij} is one-to-one, cf. [25].

3. Each variety Y_i comes with subdivision into *strata* and *cells*. Namely, we have in Y_i the collection $\mathcal{C}_i$ of subvarieties given by strict transforms of big spheres from $S(\mathcal{A}_{n-1})$ and strict transforms of all the exceptional divisors of previous blow-ups. This defines an equivalence relation on Y_i: two points x, y are equivalent if the subsets in $\mathcal{C}_i$ formed by subvarieties containing x (or respectively y) coincide. By definition, strata are equivalence classes with respect to this relation. Cells are, by definition, connected components of strata. It is clear by induction that cells are indeed topological cells and form a CW-decomposition of Y_i.

4. We define the *corner poset* of Y_i, denoted $C(Y_i)$, as follows (cf. Theorem 4.13). Its elements are pairs (Γ, Σ) where Γ is a (closure of a) cell in S^{n-2}, Σ is a stratum in Y_i such that:

(1) $q_i^{-1}(\Gamma)$ contains at least one connected component of Σ,

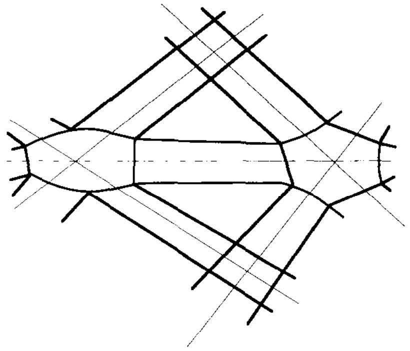


Fig. 14.

(2) $\dim(\Sigma) \geq \dim(q_i^{-1}(c)) - \dim(c)$.

We define the order on $C(Y_i)$ by setting $(\Gamma, \Sigma) \leq (\Gamma', \Sigma')$ if and only if $\Gamma' \subset \Gamma$ and $\Sigma \subset \Sigma'$.

By Theorem 4.13, the poset $C(Y_N)$ is the poset of proper faces of KP_n. On the other hand, the nerve of the poset $C(Y_0)$ is a CW-decomposition of the sphere S^{n-2} obtained by 'shrinking' all the $(n-2)$-dimensional simplices from the decomposition cut out by $S(\mathscr{A}_{n-1})$ inside themselves thus adding a new vertex at each corner (whence the name); see Fig. 14.

5. We prove by induction that the nerve of $C(Y_i)$ is homeomorphic to S^{n-2}. To do this we analyze the impact of a single blow-up on the corner poset. Namely, let Y_i be the blow-up of a subvariety $Z \subset Y_{i-1}$ which is, in its turn, the strict preimage of some geodesic sphere $G \subset S^{n-2}$-face of the configuration $S(\mathscr{A}_{n-1})$. Then Z is a closure of some stratum Z^0. For any connected component $W \subset Z^0$ the image $q_{i-1}(W)$ is a cell in S^{n-2}. The pair $(W, q_{i-1}(W))$ represents an $(n-2)$-cell of $C(Y_{i-1})$ and this cell is naturally fibered over W. We claim that $C(Y_i)$ will be obtained from $C(Y_{i-1})$ by sawing off, for each W as above, each face of $(W, q_{i-1}(W))$ parallel to W so that in the transversal section we shall have the situation depicted in Fig. 15.

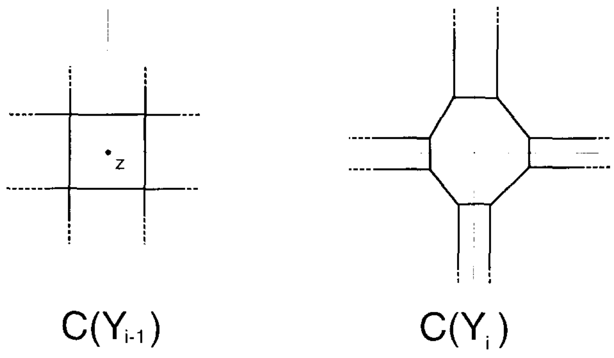


Fig. 15.

This situation is analogous to that of the theory of projective toric varieties [26]. Such varieties are classified by convex polytopes and the sawing off a face corresponds to the blow-up of a toric subvariety corresponding to this face.

But this procedure of sawing off clearly does not change the topological type of the nerve of the poset. Theorem 2.5 is proven. □

Remark. One could use as well another representation of $\overline{M_{0,n+1}}$ by blow-ups constructed in [16].

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